

Exercise 40

Show that f is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$$

Solution

The function is continuous on $(-\infty, \pi/4)$ because, assuming $a < \pi/4$,

$$\begin{aligned} \lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} \sin x \\ &= \sin \left(\lim_{x \rightarrow a} x \right) \\ &= \sin a \\ &= f(a). \end{aligned}$$

The function is continuous on $(\pi/4, \infty)$ because, assuming $a > \pi/4$,

$$\begin{aligned} \lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} \cos x \\ &= \cos \left(\lim_{x \rightarrow a} x \right) \\ &= \cos a \\ &= f(a). \end{aligned}$$

The function is continuous at $x = \pi/4$ because

$$\lim_{x \rightarrow \frac{\pi}{4}^-} f(x) = \lim_{x \rightarrow \frac{\pi}{4}^+} f(x) = f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

Therefore, f is continuous on $(-\infty, \infty)$.

